

The University of Chicago

INTERSECTOR VARIETIES IN HYPERSPACE

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1934

By

WILLIAM MARTIN BORGMAN, JR.

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INTERSECTOR VARIETIES IN HYPERSPACE

1. Introduction. A completely integrable system of linear first-order differential equations is in some situations a suitable basis upon which to build a projective theory. For example, such a system has been particularly useful in the study of varieties which are loci of linear spaces with generators in correspondence. Since two varieties of this kind play an important part in the theory of the intersector variety as it is developed in this paper, we have used a first-order system for our analytic basis. This fundamental system of first-order differential equations, together with the conditions of complete integrability, is given in Section 2. There is considerable generality in these equations, for the number of independent variables and the number of dependent variables are both quite arbitrary.

In Section 3 intersector varieties are defined geometrically in a very general manner, attention being confined to the analytic case. The dimensionality of a so-called intersection space is made quite arbitrary. Moreover, the definition of intersector variety is designed so that there remains no obvious extension of the theory, obtainable by merely increasing the dimensionality of the surrounding space, of the intersector variety itself, or of the intersection space, or by altering the division into two subsets of the points determining the ambient space. The section closes with an explanation of the fact that, when an intersector variety becomes an intersector surface or an intersector curve in ordinary space, there is no contradiction with what has already

appeared in the literature under those names.

In order to avoid trivialities and in order to make explicit certain tacit assumptions in the definition of intersector variety, it seems reasonable to put some mild limitations on four numbers, n , h , m , k , which figure in the theory and which are otherwise quite arbitrary. Section 4 is devoted to these limitations and to an explanation of why we choose to impose them.

Conditions which are both necessary and sufficient that a point θ generate an intersector variety are developed in Section 5. These conditions take the form of a set of first-order, partial or ordinary, differential equations. It seems best to carry on the study of these equations by establishing four non-overlapping cases into some one of which every set of these equations must fall. The classification is made on the basis of relative values of unity, h , and $m - k$. If the equations fall into either one of two classes, it is found that they always have solutions, and hence that the corresponding intersector varieties always exist. If they fall into either one of two other classes, they have solutions only in case integrability conditions are satisfied. Further study is restricted to one of the former pair and one of the latter pair, called Cases 1 and 2. In these cases, solutions of the equations, and hence the intersector varieties, depend only upon $m - 1$ arbitrary constants, and not upon arbitrary functions. This classification and the existence problem are discussed in Section 6.

By interchanging the rôles of the two sub-sets into which the points that determine the ambient space may be divided, a different set of intersector varieties is obtained. In Section 7, necessary and sufficient conditions that a point ψ generate an intersector variety of this kind are obtained. When the intersector

varieties of both kinds come under Cases 1 or 2, we say that they belong to associated families. By still further specialization, we find that two associated families can enter the theory symmetrically, and accordingly we call them symmetrical associated families. The latter can exist in odd spaces only. The section closes with examples of symmetrical associated families in ordinary space.

Section 8 is devoted to the derivation of the algebraic equations of the locus of so-called intersector tangent spaces at points of a fixed generator, using an appropriate local coordinate system.

In Section 9 the number of points in the first of the two sub-sets into which the $n + 1$ points are divided is limited to two, viz., x_1 and x_n . The differential equations which give the intersector varieties are found to constitute a Riccati system. A theorem is obtained to the effect that the cross ratio of the points of intersection of the line x_1x_n with four arbitrary intersector varieties of a family is independent of the position of the line x_1x_n .

From the point of view of the analysis, the limitations which are laid down in Section 4 are needlessly restrictive. In Section 10 we remove two of the restrictions on the numbers h , m , k . This gives rise to two kinds of special intersector varieties.

2. The differential equations. Our concern is with configurations which are related to $n + 1$ linearly independent points in a linear projective n -dimensional space S_n . These points will be designated by the letters $x_1, \dots, x_{n+1}$. The $n + 1$ homogeneous coordinates of any point are distinguished from one another by

superscripts. Thus the coordinates of the point x_1 are $x_1^{(1)}, \dots, x_1^{(n+1)}$. It is supposed that the coordinates of the points $x_1, \dots, x_{n+1}$ are all single-valued analytic functions of the same h independent variables $u_1, \dots, u_h$, where $0 < h \leq n$. Then each one of these points generates an h -dimensional variety, and the points of these varieties are in correspondence, the coordinates of corresponding points being given by the same values of the variables $u_1, \dots, u_h$.

Since the points $x_1, \dots, x_{n+1}$ are linearly independent, the determinant of their coordinates is not identically zero, i.e.,

$$(2.1) \quad \Delta \equiv \begin{vmatrix} x_1^{(1)} & \dots & x_{n+1}^{(1)} \\ \vdots & & \vdots \\ x_1^{(n+1)} & \dots & x_{n+1}^{(n+1)} \end{vmatrix} \equiv (x_1, \dots, x_{n+1}) \neq 0.$$

This condition makes it possible to determine the coefficients a_{ijp} in the differential equations,

$$(2.2) \quad \partial x_i / \partial u_p = \sum_{j=1}^{n+1} a_{ijp} x_j$$

$$(i = 1, \dots, n+1; p = 1, \dots, h; 0 < h \leq n),$$

whenever the coordinates of $x_1, \dots, x_{n+1}$ are known functions of the independent variables $u_1, \dots, u_h$. This is done by fixing upon a particular i and p , and then demanding that, in turn, each coordinate of the points $x_1, \dots, x_{n+1}$ satisfy the equation (2.2). The result is a set of $n+1$ linear non-homogeneous algebraic equations in the $n+1$ unknowns a_{ijp} ($j = 1, \dots, n+1$), which must have a

unique solution in view of condition (2.1).

Conversely, let us suppose that the system of differential equations (2.2) is given together with the following conditions of integrability:

$$(2.3) \quad \frac{\partial a_{ijp}}{\partial u_q} + \sum_{s=1}^{n+1} a_{isp} a_{sjq} = \frac{\partial a_{ijq}}{\partial u_p} + \sum_{s=1}^{n+1} a_{isq} a_{sjp}$$

($i, j = 1, \dots, n+1$; $p, q = 1, \dots, h$; $p < q$; $0 < h \leq n$).

These integrability conditions are sufficient to make the system (2.2) possess $n + 1$ linearly independent sets of solutions.¹ The set comprising the most general solution is a linear combination with constant coefficients of these $n + 1$ sets of particular solutions.² Therefore, these sets of particular solutions can be used as the $n + 1$ homogeneous projective coordinates of $n + 1$ linearly independent points $x_1, \dots, x_{n+1}$, and any other $n + 1$ linearly independent sets of solutions will give the coordinates of some projective transform of these points.

In this paper we shall use the first-order differential equations (2.2) subject to the integrability conditions (2.3) as the analytical basis for a projective theory.

3. Intersector varieties defined. In this section we shall define an intersector variety and then show how it may be

¹ G. M. Green, The linear dependence of functions of several variables, and completely integrable systems of homogeneous linear partial differential equations, Transactions of the American Mathematical Society, vol. 17 (1916), p. 514.

² Ibid., p. 497

specialized into an intersector curve of Lane¹ and into an intersector surface of Cook.²

Let the $n + 1$ linearly independent points $x_1, \dots, x_{n+1}$, which we have mentioned in the preceding section, and which are regarded as determining the ambient space S_n , be divided into two sets, namely, the set $x_1, \dots, x_m$ and the set $x_{m+1}, \dots, x_{n+1}$. The first set determines a linear space S_{m-1} , while the second set determines the complementary linear space S_{n-m} . As in Section 2, the coordinates of the points x_i are functions of h independent variables $u_1, \dots, u_h$. That is, each of the points has h degrees of freedom. Likewise, the linear space S_{m-1} , determined by m of the points x_i , has h degrees of freedom and therefore generates a variety of dimensionality $m + h - 1$, which we shall denote by V_{m+h-1} ; similarly, the complementary space S_{n-m} generates a variety V_{n-m+h} . Any point θ in the space S_{m-1} will generate an h -dimensional variety V_h , which will necessarily lie on the variety V_{m+h-1} . There is an h -dimensional linear space S_h which is tangent to the variety V_h at the point θ . Let us suppose that the spaces S_h and S_{n-m} intersect in a linear space of dimensionality $h - m + k$, where k is an integer subject to restrictions which are laid down in the next section, but is otherwise quite arbitrary. This linear space we shall call the intersection space and designate it by S_{h-m+k} . The locus of the point θ under these conditions will be called an intersector variety V_h of index k on the variety

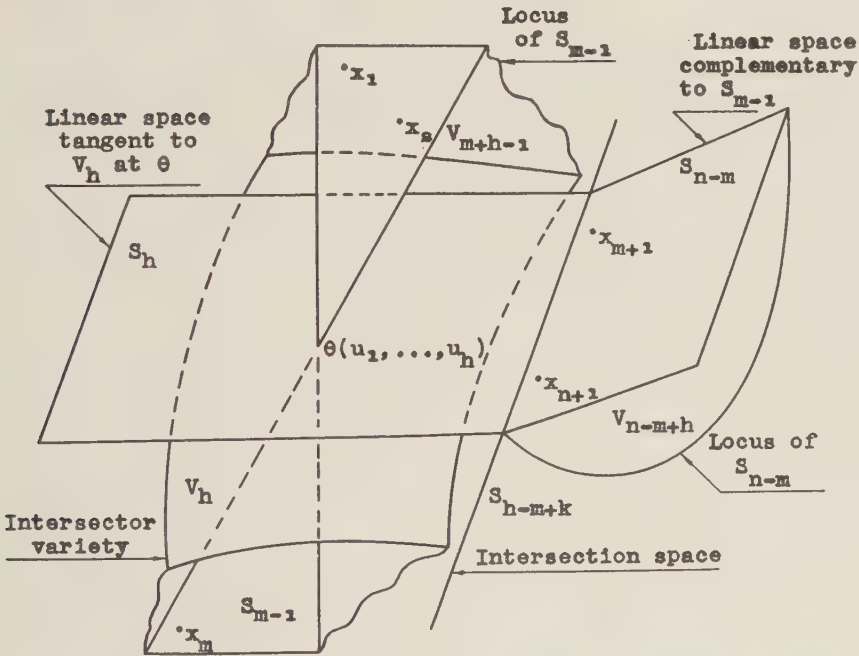
¹ E. P. Lane, Projective Differential Geometry of Curves and Surfaces, The University of Chicago Press, 1932, p. 278.

² A. J. Cook, Pairs of rectilinear congruences with generators in one-to-one correspondence, Transactions of the American Mathematical Society, vol. 32 (1930), p. 31.

V_{m+h-1} with respect to the variety V_{n-m+h} . It may be observed that k is the difference between n and the dimensionality of the ambient of the linear spaces S_h and S_{n-m} .

The following diagram, which is purely schematic, is presented in the hope that it may make it easier for the reader to comprehend the whole situation at a glance.

Surrounding space is S_n



The intersector curves of Lane are a special case of these intersector varieties. They are obtained by setting $h = 1$ and

$k = m - 1$. The original intersector curves (on a ruled surface¹) are obtained from these by setting $n = 3$ and $m = 2$.

The intersector surfaces of Cook are also a special case of the intersector variety. The definition of the latter applies to the case considered by Cook in case $n = 3$, $h = 2$, $m = 2$, $k = 1$.

In Table I we give the symbols which designate the various spaces and loci of the diagram. We also list the corresponding specialized symbols and names that apply in the special cases of intersector curves and intersector surfaces.

4. Restrictions on n , h , m , k . In the treatment of the subject of intersector varieties, we find that it is not convenient to allow any one of n , h , m , k to be zero, or to take on all positive integral values independently. Indeed, one restriction is imposed on these numbers to avoid a triviality; six restrictions are tacitly assumed in the definition of intersector variety; and three more are obtained as a consequence of some of the other restrictions. We shall now discuss each one of these limitations in turn and then close the section with a summary of all restrictions, written in a concise form.

In order to avoid a trivial situation, we require that k be different from zero. For if $k = 0$, then every variety generated by the point θ would be an intersector variety, since in S_n the spaces S_h and S_{n-m} always have a space of dimensionality $h - m$ in common. Hence we suppose

$$(4.1) \quad k \neq 0.$$

¹ E. P. Lane, Ruled surfaces with generators in one-to-one correspondence, Transactions of the American Mathematical Society, vol. 25 (1923), p. 281.

TABLE I

General		Intersector curves in S_n (Lane)			Intersector curves on a ruled surface in S_3 (Lane)		Intersector surfaces in S_3 (Cook)	
		$h-1, k=m-1$ Our notation	$h-1, m=k+1$ Lane's notation	Description	$n-3, h-1$ $m-2, k-1$	Description	$n-3, h-2$ $m-2, k-1$	Description
V_{m+h-1}	Locus of S_{m-1}	V_m	V_{k+1}	Locus of S_k	V_2	Ruled surface	V_2	Rectilinear congruence
S_{m-1}	Generator of V_{m+h-1}	S_{m-1}	S_k	Generator of V_{k+1}	S_1	Generator of the ruled surface	S_1	Generator of the congruence
V_h	Intersector variety	V_1	V_1	Intersector curve	V_1	Intersector curve on the ruled surface	V_2	Intersector surface
S_h	Intersector tangent space	S_1	S_1	Intersector tangent line	S_1	Intersector tangent line	S_2	Intersector tangent plane
S_{n-m}	Linear space complementary to S_{m-1}	S_{n-m}	S_{n-k-1}	Linear space complementary to S_k	S_1	Corr. gen. of second ruled surface	S_1	Corr. gen. of second congruence
S_{h-m+k}	Intersection space	S_0	S_0	Point common to S_1 and S_{n-k-1}	S_0	Point common to last two str. lines	S_1	Same line as the preceding

In the definition of an intersector variety, it is tacitly assumed that there exists a tangent linear space S_h . Such a space would not exist if $h = 0$, for in that event the locus of θ would be a fixed point. Therefore,

$$(4.2) \quad h \neq 0.$$

It is also tacitly assumed in the definition that k and m take on no values which lead to a contradiction of the general hypothesis that the points $x_1, \dots, x_{n+1}$ are linearly independent. If $m = k$, we are led to such a contradiction. For in this case $S_h = S_{h-m+k}$ would lie in S_{n-m} , and the point θ , necessarily in S_h , would also lie in S_{n-m} . This implies that the points $x_1, \dots, x_{n+1}$ are linearly dependent, since the spaces S_{m-1} and S_{n-m} have the point θ in common. To avoid this contradiction, we take

$$(4.3) \quad k \neq m.$$

The definition of intersector variety of course presupposes that the space S_{m-1} generates a variety which is $(m + h - 1)$ -dimensional with the point as generating element, and which is immersed in the linear space S_n . If n and h have already been chosen, then the m must be subject to the restriction

$$(4.4) \quad m + h - 1 \leq n.$$

Likewise the definition presupposes that the space S_{n-m} generates a variety which is $(n - m + h)$ -dimensional with the point as generating element. Hence

$$(4.5) \quad n - m + h \leq n.$$

The intersection space S_{h-m+k} must certainly lie in the space S_h . Therefore, we suppose that k is selected so that

$$(4.6) \quad h - m + k \leq h.$$

It is intended in the definition that k must be large enough so that the spaces S_h and S_{n-m} actually intersect. This is equivalent to the restriction

$$(4.7) \quad h - m + k \geq 0.$$

The addition of (4.4) and (4.5) gives the following limitation on h :

$$(4.8) \quad 2h \leq n + 1.$$

The inequalities (4.1), (4.3), and (4.6) imply that

$$(4.9) \quad m > 1.$$

Now combine (4.4) and (4.9) to read

$$1 < m < n - h + 2.$$

This, together with (4.2), implies that

$$0 < h < n,$$

and hence that

$$(4.10) \quad n > 1.$$

The above restrictions, which we have found it convenient to impose on n, h, m, k , are presented below in a more concise form. The numbers at the right refer to the inequalities which are used in obtaining the one written at the left.

$$\begin{aligned}
 (4.11) \quad & 1 < n && (4.10), \\
 & 0 < h < (n+3)/2 && (4.2), (4.8), \\
 & L < m < n - h + 2 && (4.4), (4.5), (4.9), \\
 & M < k < m && (4.1), (4.3), (4.6), (4.7),
 \end{aligned}$$

where L is the larger of l and $h - 1$; M is the larger of 0 and $m - h - 1$.

5. Analytic characterization of intersector varieties.

As an analytical basis for this theory, we use the fundamental differential equations (2.2). These equations determine the paths of the points x_i ($i = 1, \dots, n + 1$), and therefore they also determine the locus of the linear sub-space S_{m-1} , namely, the variety V_{m+h-1} (see the diagram, Section 3). An intersector variety V_h is the locus of the point θ , and the latter lies in the space S_{m-1} . It is clear that if the position of the point θ is determined in the space S_{m-1} , then its locus is completely determined. We propose to characterize the intersector variety V_h by obtaining an analytical expression for the position of its generating point θ in the space S_{m-1} .

Since the point θ lies in the linear space S_{m-1} , its coordinates are linear combinations of the corresponding coordinates of the points $x_1, \dots, x_m$, which determine S_{m-1} , that is,

$$\begin{aligned}
 (5.1) \quad \theta^{(1)} &= \sum_{i=1}^m \lambda_i x_i^{(1)}, \\
 &\cdot \quad \cdot \quad \cdot \\
 \theta^{(n+1)} &= \sum_{i=1}^m \lambda_i x_i^{(n+1)},
 \end{aligned}$$

where $\lambda_1, \dots, \lambda_m$ are analytic functions of the independent variables $u_1, \dots, u_h$, and where n , h , and m are restricted by the inequalities (4.11). Equations (5.1) may be replaced by the following vector equation:

$$(5.2) \quad \theta = \sum_{i=1}^m \lambda_i x_i .$$

In this equation θ and x_i are vectors, while λ_i is a scalar.

We shall first assume that the point θ generates an intersector variety, and as a consequence obtain differential equations which the λ_i must satisfy.

The linear space S_h which is tangent to the intersector variety V_h at the point θ is by definition the ambient of the tangent lines at the point θ to all curves on V_h that pass through θ . This space S_h is determined by the points¹ θ , $\partial\theta/\partial u_1, \dots, \partial\theta/\partial u_h$. Since the variety V_h is by hypothesis an intersector variety, there exist $h - m + k + 1$ linearly independent points of the tangent space S_h which lie in the space S_{n-m} . Let these points be designated by

$$(5.3) \quad \phi_s = \alpha_{s0}\theta + \sum_{p=1}^h \alpha_{sp} \partial\theta/\partial u_p \quad (s = 0, \dots, h-m+k).$$

The matrix of the coefficients α in the forms (5.3) is

¹ E. P. Lane, Projective Differential Geometry of Curves and Surfaces, p. 262.

$$(5.4) \quad \begin{pmatrix} a_{00} & a_{01} & \dots & a_{0h} \\ a_{10} & a_{11} & \dots & a_{1h} \\ . & . & . & . \\ a_{h-m+k,0} & a_{h-m+k,1} & \dots & a_{h-m+k,h} \end{pmatrix}$$

In view of the linear independence of the points ϕ_s and hence of the forms (5.3), the matrix (5.4) must be of rank¹ $h - m + k + 1$.

Since the points ϕ_s lie in the space S_{n-m} , we can write

$$(5.5) \quad \phi_s \equiv [x_{m+1}, x_{m+s}, \dots, x_{n+1}],$$

where the expression in brackets represents a linear combination of the points involved.

The coefficients a in the equality (5.3) must be such that the matrix (5.4) has rank $h - m + k + 1$, and such that equations (5.5) are valid, but they are otherwise arbitrary functions of the independent variables $u_1, \dots, u_h$.

From (5.2), (5.3), and (5.5) we have

$$a_{s0} \sum_{i=1}^m \lambda_i x_i + \sum_{p=1}^h a_{sp} \sum_{i=1}^m (x_i \partial \lambda_i / \partial u_p + \lambda_i \partial x_i / \partial u_p) \equiv [x_{m+1}, \dots, x_{n+1}]$$

$$(s = 0, \dots, h-m+k).$$

The derivatives of x_1 can be eliminated by using the fundamental

¹ L. E. Dickson, Modern Algebraic Theories, Benj. H. Sanborn and Co., 1926, p. 60.

differential equations (2,2). We then have

$$\begin{aligned} a_{s0} \sum_{i=1}^m \lambda_i x_i + \sum_{i=1}^m \left(\sum_{p=1}^h a_{sp} \partial \lambda_i / \partial u_p \right) x_i + \sum_{j=1}^{n+1} \left(\sum_{p=1}^h a_{sp} \sum_{i=1}^m \lambda_i a_{ijp} \right) x_j \\ = [x_{m+1}, \dots, x_{n+1}]. \end{aligned}$$

Since the left member cannot involve $x_1, \dots, x_m$, we equate their coefficients to zero, obtaining

$$(5.6) \quad a_{s0} \lambda_t + \sum_{p=1}^h a_{sp} (\partial \lambda_t / \partial u_p + \sum_{i=1}^m \lambda_i a_{itp}) = 0$$

$$(t = 1, \dots, m).$$

Now let us introduce the following abbreviations:

$$(5.7) \quad \begin{aligned} L_t^{(0)} &= \lambda_t, \\ L_t^{(p)} &= \partial \lambda_t / \partial u_p + \sum_{i=1}^m \lambda_i a_{itp}. \end{aligned}$$

Equations (5.6) can now be written

$$(5.8) \quad \begin{aligned} a_{00} L_t^{(0)} + a_{01} L_t^{(1)} + \dots + a_{0h} L_t^{(h)} &= 0, \\ a_{10} L_t^{(0)} + a_{11} L_t^{(1)} + \dots + a_{1h} L_t^{(h)} &= 0, \\ &\vdots \\ a_{n-m+k,0} L_t^{(0)} + \dots + a_{n-m+k,h} L_t^{(h)} &= 0. \end{aligned}$$

The matrix of the coefficients in equations (5.8) is precisely the matrix (5.4), which has rank $h - m + k + 1$. Therefore, certain $h - m + k + 1$ of the variables L_t can be found uniquely as homogeneous linear functions of the remaining $m - k$ variables.¹ Let the solutions be written as follows:

$$\begin{aligned}
 \begin{matrix} (g_0) \\ L_t \end{matrix} &= \beta_{0,1} \begin{matrix} (f_1) \\ L_t \end{matrix} + \dots + \beta_{0,m-k} \begin{matrix} (f_{m-k}) \\ L_t \end{matrix}, \\
 \begin{matrix} (g_1) \\ L_t \end{matrix} &= \beta_{1,1} \begin{matrix} (f_1) \\ L_t \end{matrix} + \dots + \beta_{1,m-k} \begin{matrix} (f_{m-k}) \\ L_t \end{matrix}, \\
 (5.9) \quad &\cdot \quad \cdot \quad \cdot \quad \cdot \\
 \begin{matrix} (g_{h-m+k}) \\ L_t \end{matrix} &= \beta_{h-m+k,1} \begin{matrix} (f_1) \\ L_t \end{matrix} + \dots + \beta_{h-m+k,m-k} \begin{matrix} (f_{m-k}) \\ L_t \end{matrix} \\
 &\quad (t = 1, \dots, m),
 \end{aligned}$$

where each β is a function of the α 's and hence of the independent variables $u_1, \dots, u_h$.

Let us consider the $(r + 1)$ th equation in (5.9) and allow the index t to take on all possible values. The following system of equations is obtained:

$$\begin{aligned}
 0 &= \beta_{r,1} \begin{matrix} (f_1) \\ L_1 \end{matrix} + \dots + \beta_{r,m-k} \begin{matrix} (f_{m-k}) \\ L_1 \end{matrix} - \begin{matrix} (g_r) \\ L_1 \end{matrix}, \\
 (5.10) \quad &\cdot \quad \cdot \quad \cdot \quad \cdot \\
 0 &= \beta_{r,1} \begin{matrix} (f_1) \\ L_m \end{matrix} + \dots + \beta_{r,m-k} \begin{matrix} (f_{m-k}) \\ L_m \end{matrix} - \begin{matrix} (g_r) \\ L_m \end{matrix}.
 \end{aligned}$$

¹ Ibid., p. 61.

Since the equations (5.10) are consistent, every $(m-k+1)$ -rowed determinant of the coefficients of the β 's must vanish,¹ that is,

$$(5.11) \quad \begin{vmatrix} (f_1) & (f_s) & \dots & (f_{m-k}) & (g_r) \\ L_q & L_q & & L_q & L_q \\ (f_1) & (f_s) & \dots & (f_{m-k}) & (g_r) \\ L_{q+1} & L_{q+1} & & L_{q+1} & L_{q+1} \\ \cdot & \cdot & & \cdot & \cdot \\ (f_1) & (f_s) & \dots & (f_{m-k}) & (g_r) \\ L_{q+m-k} & L_{q+m-k} & & L_{q+m-k} & L_{q+m-k} \end{vmatrix} = 0$$

$(q = 1, \dots, k; r = 0, \dots, h-m+k).$

Equations (5.11) follow as a consequence of the fact that the point θ , defined by equation (5.2), generates an intersector variety. Now let us consider the converse problem and accordingly assume that the functions λ_1 , which with the help of equation (5.2) define the locus of the point θ , are solutions of the equations (5.11), the functions L being defined by (5.7). With this hypothesis we shall prove that the point θ generates an intersector variety.

The equalities (5.11) imply that functions β can be found which satisfy equations (5.9). But equations (5.9) are of the form of equations (5.8) with a matrix of coefficients of rank $h - m + k + 1$, for there is one $(h - m + k + 1)$ -rowed determinant of that matrix whose value is unity. The elements of the matrix can then be used to define $h - m + k + 1$ points ϕ_s given by (5.3).

¹ Ibid., p. 61.

Furthermore, these points ϕ_s will be linearly independent, will lie in the space S_h , and will also lie in the space S_{n-m} . Therefore, S_h and S_{n-m} have a linear space S_{h-m+k} in common, and hence the variety V_h generated by the point θ is an intersector variety. This completes the proof of the

Theorem. The point θ , as defined by equation (5.2), generates an intersector variety (as shown in the diagram of Section 3) if and only if the functions λ_1 satisfy the differential equations (5.11), where the functions L_t are defined by equations (5.7).

It is understood that the letters h , m , and k have precisely the same values in equations (5.11) as they have in the diagram of Section 3. In view of the above theorem, the questions of the existence of intersector varieties, and the number of such varieties if they do exist, are intimately connected with the existence and number of solutions of the differential equations (5.11). This existence problem is discussed in the next section.

6. Existence of intersector varieties. In order to study the problem of the existence of intersector varieties and the number of such varieties when they do exist, it seems convenient to divide them into four classes according to the values of $m - k$ and h as follows:

1. $1 = m - k = h.$
2. $1 = m - k < h.$
3. $1 < m - k = h.$
4. $1 < m - k < h.$

This classification exhausts all possibilities, since, by the

last of (4.11), $m - k \leq 1$ and $m - k \neq h$. Each case will be discussed in turn.

Case 1. $1 = m - k = h$. Since there is but one independent variable in this case, the equations (5.7) are ordinary differential equations, and all the intersector varieties are curves. These necessarily include the intersector curves of Lane. The latter were explained and tabulated in Section 3. The first case also covers a type of intersector curve which Lane did not consider. It arises when $m = k + 1 = n$.

Furthermore, every intersector curve is covered by this case. For if the intersector variety is a curve, then $h = 1$, and the last of inequalities (4.11) gives us immediately $m - k = 1$.

Hence, all intersector curves and no other intersector varieties belong to Case 1.

For this case equations (5.11) become

$$(6.1) \quad \begin{vmatrix} (f_1) & (g_0) \\ L_q & L_q \end{vmatrix} = 0$$

$$\begin{vmatrix} (f_1) & (g_0) \\ L_{q+1} & L_{q+1} \end{vmatrix}$$

$$(q = 1, \dots, k).$$

Since $h = 1$, the only possible value of p in equations (5.7) is unity. Then (6.1) becomes

$$(6.2) \quad \begin{vmatrix} \lambda_q & \partial \lambda_q / \partial u_1 + \sum_{i=1}^m \lambda_i a_{i,q} \\ \lambda_{q+1} & \partial \lambda_{q+1} / \partial u_1 + \sum_{i=1}^m \lambda_i a_{i,q+1,1} \end{vmatrix} = 0$$

$$(q = 1, \dots, k).$$

Now let us replace equations (6.2) by an equivalent set. At the same time let us drop the third subscript from every a , since it is unity in each case, and use the accent mark to indicate differentiation with respect to the lone independent variable u_1 . We then have

$$(6.3) \quad \begin{vmatrix} \lambda_1 & \lambda_1' + \sum_{i=1}^m \lambda_i a_{i1} \\ \lambda_j & \lambda_j' + \sum_{i=1}^m \lambda_i a_{ij} \end{vmatrix} = 0$$

$$(j = 2, \dots, k+1).$$

Not every λ_1 can be zero, and it is no geometric restriction to suppose that λ_1 is the particular one which is not zero. With this in mind, let us expand the left member of equation (6.3), divide the equation by λ_1^2 and obtain

$$(6.4) \quad U_j' + a_{1j} - a_{11}U_j + \sum_{i=2}^{k+1} U_i(a_{ij} - U_j a_{i1}) = 0$$

$$(j = 2, \dots, k+1),$$

where $U_j = \lambda_j/\lambda_1$ and m is replaced by its equal $k+1$. Equation (6.4) is precisely the one given by Lane¹ for intersector curves on a variety V_{k+1} .

It is interesting to note that the k which we are using is the same k that Lane uses in his discussion of the intersector

¹ E. P. Lane, Projective Differential Geometry of Curves and Surfaces, p. 278.

curves. To us the k indicates the difference between n and the dimensionality of the ambient of the linear spaces S_1 and S_{n-m} (see Section 3). To Lane k indicates the dimensionality of the linear space which generates the variety on which the intersector curve is found.

The existence theorem¹ for systems of ordinary differential equations indicates that the system (6.4) has one and only one solution $U_j(u_1)$ through an initial point $(u_1^0, u_2^0, \dots, u_{k+1}^0)$. The first coordinate u_1 determines a particular generator S_k of the variety V_{k+1} ; the remaining coordinates determine the position of the point θ in that generator. Since there are ω^{m-1} points in a generator, we have the following theorem: There are ω^{m-1} intersector curves, one through each point of the generator S_k .

Case 2. $1 = m - k < h$. This is the case which covers Cook's intersector surfaces. Appropriate specialization of equations (5.11) gives

$$(6.5) \quad \begin{vmatrix} (f_1) & (g_r) \\ L_q & L_q \end{vmatrix} = 0$$

$$\begin{vmatrix} (f_1) & (g_r) \\ L_{q+1} & L_{q+1} \end{vmatrix}$$

$$(q = 1, \dots, k; r = 0, \dots, h-1).$$

For a given q there are $h + 1$ functions $L_q^{(0)}, \dots, L_q^{(h)}$ which are defined by equations (5.7). In equations (6.5) there are $h + 1$ different superscripts $f_1, g_0, \dots, g_{h-1}$. This implies that $L_q^{(0)}$

¹ Edouard Goursat, A Course in Mathematical Analysis, Vol. II, Part II of the English translation by Hedrick and Dunkel, Ginn and Co., 1917, p. 61.

must be found in at least one of the equations (6.5). Therefore, the following set is equivalent to (6.5).

$$(6.6) \quad \left| \begin{array}{c} \lambda_q \quad \partial \lambda_q / \partial u_p + \sum_{i=1}^m \lambda_i a_{i qp} \\ \lambda_{q+1} \quad \partial \lambda_{q+1} / \partial u_p + \sum_{i=1}^m \lambda_i a_{i, q+1, p} \end{array} \right| = 0$$

$$(q = 1, \dots, k; p = 1, \dots, h).$$

The equation (6.6) can be written in a form more useful for our purpose, namely,

$$(6.7) \quad \left| \begin{array}{c} \lambda_1 \quad \partial \lambda_1 / \partial u_p + \sum_{i=1}^m \lambda_i a_{i 1 p} \\ \lambda_j \quad \partial \lambda_j / \partial u_p + \sum_{i=1}^m \lambda_i a_{i j p} \end{array} \right| = 0$$

$$(j = 2, \dots, k+1; p = 1, \dots, h).$$

Expand the left member of equation (6.7), divide by λ_1^2 , which can be assumed different from zero, and obtain

$$(6.8) \quad \partial U_j / \partial u_p + a_{1 j p} - a_{1 1 p} U_j + \sum_{i=2}^m U_i (a_{i j p} - U_j a_{i 1 p}) = 0$$

$$(j = 2, \dots, m; p = 1, \dots, h),$$

where $U_j = \lambda_j / \lambda_1$, and k is replaced by its equal $m - 1$.

In order that equations (6.8) have analytic solutions,¹ it is necessary and sufficient that

$$(6.9) \quad \partial(\partial U_j / \partial u_p) / \partial u_q = \partial(\partial U_j / \partial u_q) / \partial u_p,$$

$$(j = 2, \dots, m; p, q = 1, \dots, h; p < q).$$

After a lengthy simplification which involves the integrability conditions (2.3) as well as equations (6.8), we eventually obtain the following quadratic equations in U_j which are equivalent to the conditions (6.9).

$$\begin{aligned}
 & \sum_{i=2}^m \sum_{j=m+1}^{n+1} (a_{1jq} a_{j1p} - a_{1jp} a_{j1q}) U_i U_t \\
 & - \sum_{i=2}^m \sum_{j=m+1}^{n+1} (a_{1jq} a_{jtp} - a_{1jp} a_{jtp}) U_i \\
 (6.10) \quad & + U_t \sum_{j=m+1}^{n+1} (a_{1jq} a_{j1p} - a_{1jp} a_{j1q}) \\
 & - \sum_{j=m+1}^{n+1} (a_{1jq} a_{jtp} - a_{1jp} a_{jtp}) = 0 \\
 & (t = 2, \dots, m; p, q = 1, \dots, h; p < q).
 \end{aligned}$$

The equations (6.10) must be satisfied either identically or conditionally before equations (6.8) can have solutions.

If equations (6.10) are satisfied identically, the exist-

¹ Edouard Goursat, Lecons sur l'integration des Equations aux Derivees Partielles du Premier Ordre, Librairie Scientifique, J. Hermann, 1921, p. 104.

ence theorem¹ for partial differential equations of the type to which equations (6.8) belong indicates that the system (6.8) has one and only one analytic solution $U_s(u_1, \dots, u_h), \dots, U_m(u_1, \dots, u_h)$ through an initial point $(u_1^0, \dots, u_h^0, U_s^0, \dots, U_m^0)$. The first h coordinates of this point determine a particular generator S_{m-1} on the variety V_{m+h-1} , and the remaining $m - 1$ coordinates determine the position of the point θ in that generator. Since there are ∞^{m-1} points in a generator, we have the following theorem:

Under Case 2, if the left members of (6.10) are zero for all values of the functions U_t , then there exists an $(m - 1)$ -parameter family of analytic intersector varieties on the variety V_{m+h-1} with respect to the variety V_{n-m+h} , and conversely.

Even though the integrability conditions (6.10) are not satisfied identically, it is still possible that intersector varieties exist. For any set of values of $U_s, \dots, U_m$ which satisfy both the quadratic equations (6.10) and the differential equations (6.8) will give an intersector variety.

Case 3. $1 < m - k = h$. As in the preceding case, the conditions which define this case imply that $h > 1$. That is, the intersector varieties are not curves. Let us write equations (5.11) subject to the restrictions of this case:

¹ Ibid., p. 105.

$$(6.11) \quad \begin{vmatrix} \binom{f_1}{L_q} & \binom{f_2}{L_q} & \dots & \binom{f_h}{L_q} & \binom{g_0}{L_q} \\ \binom{f_1}{L_{q+1}} & \binom{f_2}{L_{q+1}} & \dots & \binom{f_h}{L_{q+1}} & \binom{g_0}{L_{q+1}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \binom{f_1}{L_{q+h}} & \binom{f_2}{L_{q+h}} & \dots & \binom{f_h}{L_{q+h}} & \binom{g_0}{L_{q+h}} \end{vmatrix} = 0$$

$(q = 1, \dots, k).$

These equations are equivalent to equations (6.12), the left members of which are determinants that differ from one another in the last row only. The arrangement of the columns has been changed so that the superscripts now occur in natural order,

$$(6.12) \quad \begin{vmatrix} \binom{0}{L_1} & \binom{1}{L_1} & \dots & \binom{h}{L_1} \\ \cdot & \cdot & \cdot & \cdot \\ \binom{0}{L_h} & \binom{1}{L_h} & \dots & \binom{h}{L_h} \\ \binom{0}{L_{h+j}} & \binom{1}{L_{h+j}} & \dots & \binom{h}{L_{h+j}} \end{vmatrix} = 0$$

$(j = 1, \dots, k).$

If we replace the functions L_t by their values from equations (5.7), we obtain

$$\begin{array}{c}
 (6.13) \quad \left| \begin{array}{cccc}
 \lambda_1 & \partial \lambda_1 / \partial u_1 + \sum_{i=1}^m \lambda_i a_{i11} & \dots & \partial \lambda_1 / \partial u_h + \sum_{i=1}^m \lambda_i a_{i1h} \\
 \cdot & \cdot & \cdot & \cdot \\
 \lambda_h & \partial \lambda_h / \partial u_1 + \sum_{i=1}^m \lambda_i a_{ih1} & \dots & \partial \lambda_h / \partial u_h + \sum_{i=1}^m \lambda_i a_{ihh} \\
 \lambda_{j+h} & \partial \lambda_{j+h} / \partial u_1 + \sum_{i=1}^m \lambda_i a_{i,j+h,1} & \dots & \partial \lambda_{j+h} / \partial u_h + \sum_{i=1}^m \lambda_i a_{i,j+h,h}
 \end{array} \right| = 0
 \end{array}$$

($j = 1, \dots, k$).

Now let $\lambda_1, \dots, \lambda_h$ be arbitrary analytic functions of the independent variables $u_1, \dots, u_h$. Then equations (6.13) take the following normal form:

$$\begin{array}{c}
 \partial \lambda_{1+h} / \partial u_1 = F_1, \\
 \partial \lambda_{s+h} / \partial u_1 = F_s, \\
 (6.14) \quad \cdot \quad \cdot \quad \cdot \\
 \partial \lambda_m / \partial u_1 = F_{m-h},
 \end{array}$$

in which the right hand sides contain the independent variables $u_1, \dots, u_h$, the dependent variables $\lambda_{1+h}, \dots, \lambda_m$, and partial derivatives of the particular dependent variable that occurs on the

left side. The system (6.14) has analytic solutions¹. Therefore, equations (6.13) always have analytic solutions for arbitrary choice of the h functions $\lambda_1, \dots, \lambda_h$. That is, when $1 < m - k = h$, analytic intersector varieties exist and depend upon at least h arbitrary functions.

Case 4. $1 < m - k < h$. For this case the equations (5.11) are unspecialized. Obviously, integrability conditions are necessary in order to effect a solution. But those integrability conditions will not be calculated here. We state only that intersector varieties exist in Case 4 in the event that equations (5.11) are integrable.

Summary. The foregoing results are summarized in Table II.

TABLE II

Case No.	$m-k$	$h-m+k$	Analytic Intersector Varieties	
			Exist	Depend upon
1	$= 1$	$= 0$	Always	$m-1$ arbitrary constants
2	$= 1$	> 0	Subject to integrability conditions (calculated)	$m-1$ arbitrary constants
3	> 1	$= 0$	Always	At least h arbitrary functions
4	> 1	> 0	Subject to integrability conditions (not calculated)	

¹ Edouard Goursat, A Course in Mathematical Analysis, Vol. II, Part II of the English translation, p. 283.

Further discussion will be confined largely to Cases 1 and 2 where the solutions of equations (5.11) depend upon arbitrary constants and not upon arbitrary functions. Moreover, since this paper is concerned with "Intersector Varieties in Hyperspace," there will be no direct specialization of h , the dimensionality of the intersector variety V_h , or of n , the dimensionality of the space in which V_h is immersed, except possibly in mentioning special cases already in the literature. However, we feel free to vary m and k , subject to the restrictions (4.11) and to the relation $m - k = 1$, which characterizes Cases 1 and 2. With this limitation on k , the meaning previously given to k (see Section 3) can be replaced by a less awkward one, namely, the dimensionality of the linear space which generates the variety V_{m+h-1} .

7. Associated families of intersector varieties. Just as the point θ in the space S_{m-1} generates an intersector variety V_h of index k on the variety V_{m+h-1} with respect to the variety V_{n-m+h} , so the point ψ in the space S_{n-m} generates an intersector variety V'_h of index k' on the variety V_{n-m+h} with respect to the variety V_{m+h-1} . In this section the differential equations which determine the loci of the point ψ will be obtained. After this the ideas of associated families and symmetrical associated families of intersector varieties will be introduced.

In order to obtain the differential equations which determine the intersector variety V'_h , we interchange the rôles of the spaces S_{m-1} and S_{n-m} in the preceding discussion. The point ψ which generates the intersector variety V'_h is defined by the equation

$$(7.1) \quad \psi = \sum_{i=m+1}^{n+1} \lambda_i x_i .$$

Using the same method as the one used in Section 5, we find that the functions λ_i satisfy the partial differential equations

$$(7.2) \quad \begin{vmatrix} (f_1) & (f_2) & \dots & (f_{n-m-k+1}) & (g_r) \\ L_q & L_q & & L_q & L_q \\ (f_1) & (f_2) & \dots & (f_{n-m-k+1}) & (g_r) \\ L_{q+1} & L_{q+1} & & L_{q+1} & L_{q+1} \\ . & . & . & . & . \\ (f_1) & (f_2) & \dots & (f_{n-m-k+1}) & (g_r) \\ L_{q+n-m-k+1} & L_{q+n-m-k+1} & & L_{q+n-m-k+1} & L_{q+n-m-k+1} \end{vmatrix} = 0$$

$$(q = m+1, \dots, m+k; r = 0, \dots, h+m+k-n-1),$$

where the functions L are defined by

$$(7.3) \quad \begin{aligned} L_t^{(0)} &= \lambda_t, \\ L_t^{(p)} &= \partial \lambda_t / \partial u_p + \sum_{i=m+1}^{n+1} \lambda_i a_{it} p \\ (t &= m+1, \dots, n+1; p = 1, \dots, h). \end{aligned}$$

The complete result is stated in the following

Theorem. The point ψ as defined by equation (7.1) generates an intersector variety V'_h of index k' on the variety V_{n-m+h} with respect to the variety V_{m+h-1} if and only if the func-

tions λ_i satisfy the differential equations (7.2), where the functions L_t are given by (7.3).

The inequalities which correspond to (4.11) are

$$(7.4) \quad \begin{aligned} 1 &< n, \\ 0 &< h < (n+3)/2, \\ h-1 &< m < S, \\ M' &< k' < n-m+1, \end{aligned}$$

where S is the smaller of n and $n - h + 2$, and M' is the larger of 0 and $n - m - h$.

It was indicated at the close of the last section that Cases 1 and 2 would be given greater consideration than the other two cases. In accordance with this plan, let

$$(7.5) \quad k = m - 1, \quad k' = n - m,$$

which means that the points θ and ψ both generate intersector varieties belonging to Case 1 or Case 2. If equations (7.5) are valid, the $(m - 1)$ -parameter family to which the intersector varieties V_h belong and the $(n - m)$ -parameter family to which the intersector varieties V_h' belong will be known as associated families of intersector varieties.

For convenience of reference we combine equations (6.4) and (6.8), and specialize equation (7.2) by making $k' = n - m$. This gives the differential equations of associated families of intersector varieties.

$$\theta = \sum_{i=1}^m \lambda_i x_i ,$$

$$\partial U_j / \partial u_p + a_{1jp} - a_{11p} U_j + \sum_{i=2}^m U_i (a_{1ip} - U_j a_{i1p}) = 0$$

$$(7.6) \quad (j = 2, \dots, m; p = 1, \dots, h),$$

$$\psi = \sum_{i=m+1}^{n+1} \lambda_i x_i ,$$

$$\partial U_s / \partial u_{p+m+1,s,p} - a_{m+1,m+1,p} U_s + \sum_{i=m+2}^{n+1} U_i (a_{isp} - U_s a_{i,m+1,p}) = 0$$

$$(s = m+2, \dots, n+1; p = 1, \dots, h),$$

where

$$U_j = \lambda_j / \lambda_1, \quad U_s = \lambda_s / \lambda_{m+1} .$$

If the intersector varieties of both of the associated families depend upon the same number of arbitrary constants, then

$$(7.7) \quad n = 2m - 1.$$

The result is a symmetrical situation in which $k' = k$. Let two associated families of intersector varieties for which $k' = k$ be called symmetrical associated families of intersector varieties. In view of equation (7.7), symmetrical associated families of intersector varieties are found only in linear spaces of odd dimensions.

If $m = 2$ and $h = 1$, we have two symmetrical associated families of intersector curves in ordinary space. The curves of each family are found on one of two ruled surfaces with generators in correspondence, but non-intersecting. The case was first discussed by Lane¹ in 1923.

If $m = 2$ and $h = 2$, we have two symmetrical associated families of intersector surfaces in ordinary space. This case was discussed by Cook² in a paper previously cited.

8. Locus of intersector tangent spaces. In Cases 1 and 2, the algebraic equations of the variety which is the locus of the intersector tangent spaces S_h at points θ of a fixed generator S_{m-1} are readily obtained if its dimensionality is less than n and if a local coordinate system is suitably chosen. To this end let us select the points $x_1, \dots, x_{n+1}$ for the vertices of a local pyramid of reference, with the unit point chosen so that the point ϕ will have local coordinates proportional to $\phi_1, \dots, \phi_{n+1}$. Then if ϕ is any point in the tangent linear space S_{m-1} , we have

$$(8.1) \quad \phi = \alpha_0 \theta + \sum_{s=1}^h \alpha_s \partial \theta / \partial u_s \equiv \phi_1 x_1 + \dots + \phi_{n+1} x_{n+1}.$$

The function θ and its derivatives can be eliminated from (8.1) by using the first two equations of (7.6). This gives

¹ E. P. Lane, Ruled surfaces with generators in one-to-one correspondence, Transactions of the American Mathematical Society, vol. 25 (1923), p. 287.

² Loc. cit., p. 38.

$$\begin{aligned}
 \lambda_1 \phi_r &= \lambda_r \phi_1 & (r = 2, \dots, m), \\
 (8.2) \quad \rho \phi_t &= \sum_{s=1}^h \alpha_s \sum_{i=1}^m \lambda_i a_{its} & (t = m+1, \dots, n+1),
 \end{aligned}$$

where ρ is a factor of proportionality. By replacing λ_1 in the second of (8.2) by its value from the first of (8.2) and then eliminating ρ and α_s from the result, we have

$$\begin{aligned}
 (8.3) \quad & \left| \begin{array}{cccc}
 \phi_t & \sum_{i=1}^m \phi_i a_{it1} & \sum_{i=1}^m \phi_i a_{its} & \dots & \sum_{i=1}^m \phi_i a_{ith} \\
 \phi_{t+1} & \sum_{i=1}^m \phi_i a_{i,t+1,1} & \sum_{i=1}^m \phi_i a_{i,t+1,s} & \dots & \sum_{i=1}^m \phi_i a_{i,t+1,h} \\
 \cdot & \cdot & \cdot & \cdot & \cdot \\
 \phi_{t+h} & \sum_{i=1}^m \phi_i a_{i,t+h,1} & \sum_{i=1}^m \phi_i a_{i,t+h,s} & \dots & \sum_{i=1}^m \phi_i a_{i,t+h,h}
 \end{array} \right| = 0 \\
 & (t = m+1, \dots, n-h+1).
 \end{aligned}$$

Equations (8.3) are the equations of the locus of the intersector tangent spaces S_h at points θ of a fixed generator S_{m-1} . Since the dimensionality of this locus is less than n , then $(m-1) + h < n$, and hence $m+1 \leq n-h+1$. Therefore, equations (8.3) are always obtainable.

In case $n \geq m(h+1)$, then $n+1-m(h+1)$ of the $(h+1)$ th degree equations (8.3) can be replaced by the following linear equations:

$$(8.4) \quad \begin{vmatrix} \phi_t & a_{1t1} & \cdots & a_{mth} \\ \phi_{t+1} & a_{1,t+1,1} & \cdots & a_{m,t+1,h} \\ \cdot & \cdot & \cdot & \cdot \\ \phi_{t+mh} & a_{1,t+mh,1} & \cdots & a_{m,t+mh,h} \end{vmatrix} = 0$$

$$(t = m+1, \dots, n+1-mh).$$

The equations of the variety which is the locus of the tangent spaces S_h to the intersector varieties of the associated family may be obtained in the same manner from the second two equations of (7.6).

9. Special case: $m = 2$. In this case there is a one-parameter family of intersector varieties. These are generated by a one-parameter family of points θ that lie on a straight line. In other words, corresponding points on all intersector varieties of the family lie on the same straight line.

Let us combine equations (6.4) and (6.8), making $m = 2$. We then have

$$(9.1) \quad \partial U_s / \partial u_p = -a_{1sp} + (a_{11p} - a_{s2p})U_s + a_{s1p}U_s^2$$

$$(p = 1, \dots, h).$$

Now let $p = 1$ and regard U_s as the dependent variable, u_1 as the independent variable, and $u_2, \dots, u_h$ as parameters. The equation is an equation of Riccati, and hence the cross ratio of any four

particular solutions is independent of u_1 . Similarly, if $p = t$, the cross ratio of any four particular values of U_s which satisfy the equation is independent of u_t . We assume that the integrability conditions for the system (8.1) are satisfied identically so that the system is completely integrable. Therefore, there is a one-parameter family of solutions $U_s = U_s(u_1, \dots, u_h)$. But they have the property that the cross ratio of four particular solutions is independent of $u_1, \dots, u_h$ and is therefore constant. This leads to the

Theorem: Any four intersector varieties V_h on V_{h+1} with respect to V_{n+h-s} intersect each rectilinear generator x_1x_s of V_{h+1} in four points whose cross ratio is independent of the generator.

This theorem is an obvious extension of one given by Lane¹ for intersector curves on a ruled surface.

10. Special intersector varieties. The analytical work of Sections 5 and 6 can be applied to two kinds of special intersector varieties.

The first arises if we remove the restriction (4.4) and on the contrary require that

$$(10.1) \quad h + m - n > 1.$$

Geometrically this means that the space S_{m-1} generates a variety which is n -dimensional with the point as generating element, whatever the value of h may be, as long as it satisfies (10.1).

¹ E. P. Lane, Ruled surfaces with generators in one-to-one correspondence, Transactions of the American Mathematical Society, vol. 25 (1923), p. 289.

Now the intersection space S_{h-m+k} lies in the space S_{n-m} and therefore

$$(10.2) \quad m - k \geq h + m - n.$$

Hence, by (10.1) and (10.2), we have $m - k > 1$, and therefore special intersector varieties of this kind belong to Cases 3 or 4.

The second kind of special intersector variety arises if we remove the restriction (4.5) and on the contrary require that

$$(10.3) \quad m < h.$$

Geometrically this means that the space S_{n-m} generates a variety which is n -dimensional with the point as generating element, whatever the value of h may be, as long as it satisfies (10.3). But (4.9) and (10.3) imply $h > 2$, and hence that special intersector varieties of this kind are neither curves nor surfaces.

VITA

William Martin Borgman, Jr. was born November 20, 1903 in Detroit, Michigan. His secondary school and junior college work were done in the public schools of that city. He received the degree of Bachelor of Science in Engineering from the University of Michigan in 1924 and the degree of Master of Science from the University of Chicago in 1931. Since 1925 he has been an instructor in mathematics at Wayne University, Detroit, Michigan. During the summer quarters of 1926-29 and 1931-34, and throughout the academic year of 1930-31 he was in residence at the University of Chicago. Here he studied under Professors Albert, Barnard, Bartky, Bliss, Dickson, Graves, Lane, Logsdon, MacMillan, and Slaughter and under visiting Professors Bell, Bol, Coble, Hille, Jackson, Mitchell, and Murnaghan.

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